Monadic Abstract Interpreters

Ilya Sergey   Dominique Devriese   Matthew Might

Jan Midtgaard   David Darais   Dave Clarke   Frank Piessens

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"My life goal: Replace myself with a $\LaTeX$ macro."

Matthew Might

Abstract Interpreters for Free, SAS 2010

M. Might, *“Abstract Interpreters for Free”*

small-step concrete semantics (interpreter)

=>

small-step abstract semantics (analysis)

This Work

This Work

Replace myself with a library of reusable functions.

This Work

small-step concrete semantics implementation

and

small-step abstract semantics implementation

This Work

small-step concrete semantics implementation

and

small-step abstract semantics implementation

(for the price of one + $\mathcal{E}$)

How do you

design

an abstract interpreter?

How do you

implement

an abstract interpreter?

Our perspective

Separate
the interpreter machinery
from a program analysis logic

Separate
the interpreter machinery
from a program analysis logic

and also

Make different aspects of a program analysis
reusable between languages and semantics

Monads for
the separation of concerns

Starting point: Concrete vs. Abstract

Concrete CPS semantics

$$
\begin{aligned}
([\![(f\ æ_1 \ldots æ_n)]\!], \rho, \sigma) &\Rightarrow (call, \rho'', \sigma'), \text{ where} \\
([\![(\lambda\ (v_1 \ldots v_n)\ call)]\!], \rho') &= \mathcal{A}(f, \rho, \sigma) \\
\rho'' &= \rho'[v_i \mapsto a_i] \\
\sigma' &= \sigma[a_i \mapsto \mathcal{A}(æ_i, \rho, \sigma)] \\
a_i &= alloc(v_i, \sigma)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}(v, \rho, \sigma) &= \sigma(\rho(v)) \\
\mathcal{A}(lam, \rho, \sigma) &= (lam, \rho)
\end{aligned}
$$

Abstract CPS semantics

$$
\begin{aligned}
([\![(f\ æ_1 \ldots æ_n)]\!], \hat{\rho}, \hat{\sigma}) &\rightsquigarrow (call, \hat{\rho}'', \hat{\sigma}'), \text{ where} \\
([\![(\lambda\ (v_1 \ldots v_n)\ call)]\!], \hat{\rho}') &\in \hat{\mathcal{A}}(f, \hat{\rho}, \hat{\sigma}) \\
\hat{\rho}'' &= \hat{\rho}'[v_i \mapsto \hat{a}_i] \\
\hat{\sigma}' &= \hat{\sigma} \sqcup [\hat{a}_i \mapsto \hat{\mathcal{A}}(æ_i, \hat{\rho}, \hat{\sigma})] \\
\hat{a}_i &= \widehat{alloc}(v_i, \hat{\sigma})
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\mathcal{A}}(v, \hat{\rho}, \hat{\sigma}) &= \hat{\sigma}(\hat{\rho}(v)) \\
\hat{\mathcal{A}}(lam, \hat{\rho}, \hat{\sigma}) &= \{(lam, \hat{\rho})\}
\end{aligned}
$$

Similar, but not the same!

$$
\begin{aligned}
(\llbracket(f\ æ_1 \ldots æ_n)\rrbracket, \rho, \sigma) &\Rightarrow (call, \rho'', \sigma'), \text{ where} \\
(\llbracket(\lambda\ (v_1 \ldots v_n)\ call)\rrbracket, \rho') &= \mathcal{A}(f, \rho, \sigma) \\
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a_i &= alloc(v_i, \sigma)
\end{aligned}
$$

$$
\begin{aligned}
(\llbracket(f\ æ_1 \ldots æ_n)\rrbracket, \hat{\rho}, \hat{\sigma}) &\leadsto (call, \hat{\rho}'', \hat{\sigma}'), \text{ where} \\
(\llbracket(\lambda\ (v_1 \ldots v_n)\ call)\rrbracket, \hat{\rho}') &\in \hat{\mathcal{A}}(f, \hat{\rho}, \hat{\sigma}) \\
\hat{\rho}'' &= \hat{\rho}'[v_i \mapsto \hat{a}_i] \\
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\hat{a}_i &= \widehat{alloc}(v_i, \hat{\sigma})
\end{aligned}
$$

How can we unify
their implementations?

Commonalities

Differences

Commonalities

Commonalities

Shape of the computation

Concrete $\overbrace{([\![(f\ æ_1 \ldots æ_n)]\!], \rho, \sigma)}^{\varsigma} \Rightarrow (call, \rho'', \sigma')$

Abstract $\overbrace{([\![(f\ æ_1 \ldots æ_n)]\!], \hat{\rho}, \hat{\sigma})}^{\hat{\varsigma}} \leadsto (call, \hat{\rho}'', \hat{\sigma}')$

Differences

Differences

Treatment of semantic values

Concrete

$$
\begin{aligned}
([\![(f\ æ_1 \ldots æ_n)]\!], \rho, \sigma) &\Rightarrow (call, \rho'', \sigma'), \text{ where} \\
([\![(\lambda\ (v_1 \ldots v_n)\ call)]\!], \rho') &= \mathcal{A}(f, \rho, \sigma) \\
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Abstract

$$
\begin{aligned}
([\![(f\ æ_1 \ldots æ_n)]\!], \hat{\rho}, \hat{\sigma}) &\leadsto (call, \hat{\rho}'', \hat{\sigma}'), \text{ where} \\
([\![(\lambda\ (v_1 \ldots v_n)\ call)]\!], \hat{\rho}') &\in \hat{\mathcal{A}}(f, \hat{\rho}, \hat{\sigma}) \\
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Differences

Treatment of semantic values

Concrete

$$([\![(f\ æ_1 \ldots æ_n)]\!], \rho, \sigma) \Rightarrow (call, \rho'', \sigma'), \text{ where}$$
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$$a_i = alloc(v_i, \sigma)$$

Abstract

$$([\![(f\ æ_1 \ldots æ_n)]\!], \hat{\rho}, \hat{\sigma}) \leadsto (call, \hat{\rho}'', \hat{\sigma}'), \text{ where}$$
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$$\hat{a}_i = \widehat{alloc}(v_i, \hat{\sigma})$$

Abstract Interpreter

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- Forks

Abstract Interpreter

- Forks
- Advances timestamps

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- Forks
- Advances timestamps
- Performs Abstract GC

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- Forks
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- Keeps track of contexts

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- Keeps track of contexts
- Makes counting

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} Computational Effects

Abstract Interpreter

nondeterminism
- Forks

tracing
- Advances timestamps

state modification
- Performs Abstract GC

tracing
- Keeps track of contexts

state modification
- Makes counting

} Computational Effects

Abstract Interpretation as a computational effect

Eugenio Moggi

Notions of computation and monads, *Inf. Comput.*, 1991

... we identify the type *A* with the object of *values* (of type *A*) and obtain the object of *computations* (of type *A*) by applying an unary type-constructor *T* to *A*.
We call *T* a *notion of computation*, since it abstracts away from the type of values computations may produce.

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Philip Wadler

Comprehending Monads, LFP, 1991

It is relatively straightforward to adopt Moggi's technique of structuring denotational specifications into a technique for structuring functional programs. This paper presents a simplified version of Moggi's ideas, framed in a way better suited to functional programmers than semanticists; in particular, no knowledge of category theory is assumed.

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It is relatively straightforward to adopt Moggi's technique of structuring denotational specifications into a technique for structuring functional programs. This paper presents a simplified version of Moggi's ideas, framed in a way better suited to functional programmers than semanticists; in particular, no knowledge of category theory is assumed.

Let’s program some semantics in Haskell

Implementing and Refactoring time-stamped *k*-CFA for CPS

$$v \in \mathsf{Var} \text{ is a set of identifiers}$$

$$lam \in \mathsf{Lam} ::= (\lambda\ (v_1 \ldots v_n)\ call)$$

$$f, æ \in \mathsf{AExp} = \mathsf{Var} + \mathsf{Lam}$$

$$call \in \mathsf{Call} ::= (f\ æ_1 \ldots æ_n) + \mathsf{Exit}$$

$$\hat{\varsigma} \in \hat{\Sigma} = \mathsf{Call} \times \widehat{Env} \times \widehat{Store} \times \widehat{Time}$$

$$\hat{\rho} \in \widehat{Env} = \mathsf{Var} \rightharpoonup \widehat{Addr}$$

$$\hat{\sigma} \in \widehat{Store} = \widehat{Addr} \to \mathcal{P}(\hat{D})$$

$$\hat{d} \in \hat{D} = \widehat{Clo}$$

$$\widehat{clo} \in \widehat{Clo} = \mathsf{Lam} \times \widehat{Env}$$

$$\hat{a} \in \widehat{Addr} = \mathsf{Var} \times \widehat{Time}$$

$$\hat{t} \in \widehat{Time} = \mathsf{Call}^{\leq k}$$

$$
\begin{array}{llll}
\textbf{type}\ \mathit{Var} & = \mathit{String} & & \\
\textbf{data}\ \mathit{Lambda} & = [\,\mathit{Var}\,] \Rightarrow \mathit{CExp} & \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}) & \\
\textbf{data}\ \mathit{AExp} & = \mathit{Ref}\ \ \mathit{Var} & & \\
 & \mid\ \ \mathit{Lam}\ \mathit{Lambda} & \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}) & \\
\textbf{data}\ \mathit{CExp} & = \mathit{Call}\ \mathit{AExp}\ [\,\mathit{AExp}\,] & & \\
 & \mid\ \ \mathit{Exit} & \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}) &
\end{array}
$$

$$
\begin{aligned}
\hat{\varsigma} \in \hat{\Sigma} &= \mathsf{Call} \times \widehat{\mathit{Env}} \times \widehat{\mathit{Store}} \times \widehat{\mathit{Time}} \\
\hat{\rho} \in \widehat{\mathit{Env}} &= \mathsf{Var} \rightharpoonup \widehat{\mathit{Addr}} \\
\hat{\sigma} \in \widehat{\mathit{Store}} &= \widehat{\mathit{Addr}} \to \mathcal{P}(\hat{D}) \\
\hat{d} \in \hat{D} &= \widehat{\mathit{Clo}} \\
\widehat{\mathit{clo}} \in \widehat{\mathit{Clo}} &= \mathsf{Lam} \times \widehat{\mathit{Env}} \\
\hat{a} \in \widehat{\mathit{Addr}} &= \mathsf{Var} \times \widehat{\mathit{Time}} \\
\hat{t} \in \widehat{\mathit{Time}} &= \mathsf{Call}^{\leq k}
\end{aligned}
$$

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\textbf{type}\ \mathit{Var} & = \mathit{String} & & \\
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\textbf{data}\ \mathit{AExp} & = \mathit{Ref}\ \ \mathit{Var} & & \\
 & \mid\ \ \mathit{Lam}\ \mathit{Lambda} & \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}) & \\
\textbf{data}\ \mathit{CExp} & = \mathit{Call}\ \mathit{AExp}\ [\mathit{AExp}] & & \\
 & \mid\ \ \mathit{Exit} & \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}) &
\end{array}
$$

$$
\begin{array}{ll}
\textbf{type}\ \Sigma & = (\mathit{CExp}, \mathit{Env}, \mathit{Store}, \mathit{Time}) \\
\textbf{type}\ k \rightharpoonup v & = \mathit{Map}\ k\ v \\
\textbf{type}\ \mathit{Env} & = \mathit{Var} \rightharpoonup \mathit{Addr} \\
\textbf{type}\ \mathit{Store} & = \mathit{Addr} \rightharpoonup \mathcal{P}\ \mathit{Val} \\
\textbf{data}\ \mathit{Val} & = \mathit{Clo}\ (\mathit{Lambda}, \mathit{Env}) \\
 & \qquad\quad \textbf{deriving}\ (\mathit{Eq}, \mathit{Ord}) \\
\textbf{type}\ \mathit{Addr} & = (\mathit{Var}, \mathit{Time}) \\
\textbf{type}\ \mathit{Time} & = [\mathit{CExp}]
\end{array}
$$

$$(\leadsto) \in \Sigma \to \mathcal{P}(\Sigma)$$

$$\overbrace{(\llbracket (f\ æ_1 \ldots æ_n) \rrbracket, \hat{\rho}, \hat{\sigma}, \hat{t})}^{\hat{\varsigma}} \leadsto (call, \hat{\rho}'', \hat{\sigma}', \hat{t}'), \text{ if}$$

$$\underbrace{(\llbracket (\lambda\ (v_1 \ldots v_n)\ call) \rrbracket, \hat{\rho}')}_{\widehat{clo}} \in \hat{\mathcal{A}}(f, \hat{\rho}, \hat{\sigma})$$

$$\hat{t}' = \widehat{tick}(\widehat{clo}, \hat{\varsigma})$$

$$\hat{a}_i = \widehat{alloc}(v_i, \hat{t}')$$

$$\hat{d}_i \in \hat{\mathcal{A}}(æ_i, \hat{\rho}, \hat{\sigma})$$

$$\hat{\rho}'' = \hat{\rho}'[v_i \mapsto \hat{a}_i]$$

$$\hat{\sigma}' = \hat{\sigma} \sqcup [\hat{a}_i \mapsto \{\hat{d}_i\}]$$

$$next :: \Sigma \to [\Sigma]$$

$$\overbrace{(\llbracket(f\ æ_1 \ldots æ_n)\rrbracket, \hat{\rho}, \hat{\sigma}, \hat{t})}^{\hat{\varsigma}} \rightsquigarrow (call, \hat{\rho}'', \hat{\sigma}', \hat{t}'), \text{ if}$$

$$\underbrace{(\llbracket(\lambda\ (v_1 \ldots v_n)\ call)\rrbracket, \hat{\rho}')}_{\widehat{clo}} \in \hat{\mathcal{A}}(f, \hat{\rho}, \hat{\sigma})$$

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$$next :: \Sigma \to [\Sigma]$$

$$\begin{aligned}
next\ \varsigma@(Call\ f\ \ aes, \rho, \sigma, t) = [\,(call, \rho'', \sigma', t')\ |\\
&proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow Set.toList\ (arg\ (f, \rho, \sigma)),\\
&\mathbf{let}\ t' \ = tick\ (proc, \varsigma)\\
&\qquad as \ = [\,alloc\ (v, t', proc, \varsigma)\ |\ v \leftarrow vs\,]\\
&\qquad ds \ = [\,arg\ (æ, \rho, \sigma)\ |\ æ \leftarrow aes\,]\\
&\qquad \rho'' = \rho' \,/\!/\, [\,v \Longrightarrow a\ |\ v \leftarrow vs\ |\ a \leftarrow as\,]\\
&\qquad \sigma' \ = \sigma \sqcup [\,a \Longrightarrow d\ |\ a \leftarrow as\ |\ d \leftarrow ds\,]]
\end{aligned}$$

$$next\ \varsigma = [\varsigma]$$

Refactoring Plan

- Capture non-determinism in the monad
- Pull the store into the monad
- Pull the time into the monad
- Abstract over *k*-CFA addresses

Refactoring Plan

➔ • Capture non-determinism in the monad

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$$next :: \Sigma \to [\Sigma]$$

$$\begin{aligned}
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&\qquad proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow Set.toList\ (arg\ (f, \rho, \sigma)), \\
&\qquad \mathbf{let}\ t' \ = tick\ (proc, \varsigma) \\
&\qquad\qquad as \ = [\,alloc\ (v, t', proc, \varsigma) \mid v \leftarrow vs] \\
&\qquad\qquad ds \ = [\,arg\ (\text{æ}, \rho, \sigma) \mid \text{æ} \leftarrow aes] \\
&\qquad\qquad \rho'' = \rho' \mathbin{/\!/} [\,v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as] \\
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\end{aligned}$$

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$$mnext :: \Sigma \to [\Sigma]$$

$$\begin{aligned}
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\end{aligned}$$

$$next\ \varsigma = [\varsigma]$$

$$mnext :: \Sigma \rightarrow [\Sigma]$$

$$\begin{aligned}
&mnext\ \varsigma@(Call\ f\ \ aes, \rho, \sigma, t) = \mathbf{do}\\
&\quad proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow Set.toList\ (arg\ (f, \rho, \sigma)),\\
&\quad \mathbf{let}\ t' \ = tick\ (proc, \varsigma)\\
&\qquad as\ = [\,alloc\ (v, t', proc, \varsigma) \mid v \leftarrow vs\,]\\
&\qquad ds\ = [\,arg\ (\text{æ}, \rho, \sigma) \mid \text{æ} \leftarrow aes\,]\\
&\qquad \rho'' = \rho' \mathbin{/\!/} [\,v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as\,]\\
&\qquad \sigma'\ = \sigma \sqcup [\,a \Longrightarrow d \mid a \leftarrow as \mid d \leftarrow ds\,]\\
&\quad return\ (call, \rho'', \sigma', t')
\end{aligned}$$

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\end{aligned}
$$

$$mnext\ \varsigma = return\ \varsigma$$

Semantic functions

$$
\begin{aligned}
fun &:: (Env, Store) \to AExp \to [\,Val\,] \\
arg &:: (Env, Store) \to AExp \to [\,Val\,] \\
tick &:: Val \to State \to [\,Time\,] \\
alloc &:: (\,Time, Val, State) \to Var \to [\,Addr\,]
\end{aligned}
$$

Semantic functions

$$\left.\begin{array}{l} \mathit{fun} \;\; :: (\mathit{Env}, \mathit{Store}) \to \mathit{AExp} \to [\,\mathit{Val}\,] \\ \mathit{arg} \;\; :: (\mathit{Env}, \mathit{Store}) \to \mathit{AExp} \to [\,\mathit{Val}\,] \end{array}\right\} \quad \hat{\mathcal{A}}(f, \hat{\rho}, \hat{\sigma})$$

$$\mathit{tick} \;\; :: \mathit{Val} \to \mathit{State} \to [\,\mathit{Time}\,]$$

$$\mathit{alloc} :: (\mathit{Time}, \mathit{Val}, \mathit{State}) \to \mathit{Var} \to [\,\mathit{Addr}\,]$$

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\end{aligned}
$$

$$mnext\ \varsigma = return\ \varsigma$$

$$mnext :: \Sigma \to [\Sigma]$$

$$
\begin{aligned}
&mnext\ \varsigma@(Call\ f\ \ aes, \rho, \sigma, t) = \mathbf{do}\\
&\quad proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow\ fun\ (\rho, \sigma)\ f\\
&\quad t' \leftarrow tick\ proc\ \varsigma\\
&\quad \mathbf{let}\ as = mapM\ \ (alloc\ (t', proc, \varsigma))\ vs\\
&\quad\qquad ds = mapM\ \ (arg\ (\rho, \sigma))\ aes\\
&\quad\qquad \rho'' = \rho' /\!/ [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as]\\
&\quad\qquad \sigma'\ = \sigma \sqcup [a \Longrightarrow d \mid a \leftarrow as \mid d \leftarrow ds]\\
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\end{aligned}
$$

$$mnext\ \varsigma = return\ \varsigma$$

Refactoring Plan

➔ • Capture non-determinism in the monad
- Pull the store into the monad
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\end{aligned}
$$

Semantic functions

$$
\begin{aligned}
&\textbf{class } Monad\ m \Rightarrow CPSInterface\ m \textbf{ where}\\
&\quad fun \quad :: Env \to AExp \to m\ Val\\
&\quad arg \quad :: Env \to AExp \to m\ Val\\
&\quad (\mapsto) \ :: Addr \to Val \to m\ ()\\
&\quad alloc :: Time \to Var \to m\ Addr\\
&\quad tick \quad :: Val \to P\Sigma \to m\ Time
\end{aligned}
$$

Semantic functions

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&\quad (\mapsto)\ :: \mathit{Addr} \to \mathit{Val} \to m\ ()\\
&\quad \mathit{alloc} :: \mathit{Time} \to \mathit{Var} \to m\ \mathit{Addr}\\
&\quad \mathit{tick}\ \ :: \mathit{Val} \to P\Sigma \to m\ \mathit{Time}
\end{aligned}
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\quad fun\ \ :: Env \to AExp \to m\ Val \\
\quad arg\ \ :: Env \to AExp \to m\ Val \\
\quad (\mapsto)\ :: Addr \to Val \to m\ () \\
\quad alloc :: Time \to Var \to m\ Addr \\
\quad tick\ \ :: Val \to P\Sigma \to m\ Time
\end{array}
$$

$\underbrace{(CExp, Env, Time)}$ - "partial state"

Semantic Interface

$$\begin{array}{l}
\textbf{class}\ Monad\ m \Rightarrow CPSInterface\ m\ \textbf{where} \\
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\quad alloc :: Time \to Var \to m\ Addr \\
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\end{array}$$

$\overbrace{(CExp, Env, Time)}$ - "partial state"

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&\quad t' \leftarrow tick\ proc\ \varsigma \\
&\quad \mathbf{let}\ as = mapM\ (alloc\ (t', proc, \varsigma))\ vs \\
&\quad\quad\quad ds = mapM\ (arg\ (\rho, \sigma))\ aes \\
&\quad\quad\quad \rho'' = \rho' \mathbin{/\!/} [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as] \\
&\quad\quad\quad \sigma' = \sigma \sqcup [a \Longrightarrow d \mid a \leftarrow as \mid d \leftarrow ds] \\
&\quad return\ (call, \rho'', \sigma', t')
\end{aligned}
$$

$$mnext\ \varsigma = return\ \varsigma$$

$$mnext :: (CPSInterface\ m) \Rightarrow P\Sigma \to m\ P\Sigma$$

$$
\begin{aligned}
& mnext\ \varsigma@(Call\ f\ aes, \rho, \sigma, t) = \mathbf{do} \\
& \quad proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow fun\ (\rho, \sigma)\ f \\
& \quad t' \leftarrow tick\ proc\ \varsigma \\
& \quad \mathbf{let}\ as = mapM\ (alloc\ (t', proc, \varsigma))\ vs \\
& \quad \phantom{\mathbf{let}}\ ds = mapM\ (arg\ (\rho, \sigma))\ aes \\
& \quad \phantom{\mathbf{let}}\ \rho'' = \rho' \mathbin{/\!/} [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as] \\
& \quad \phantom{\mathbf{let}}\ \sigma' = \sigma \sqcup [a \Longrightarrow d \mid a \leftarrow as \mid d \leftarrow ds] \\
& \quad return\ (call, \rho'', \sigma', t')
\end{aligned}
$$

$$mnext\ \varsigma = return\ \varsigma$$

$$mnext :: (CPSInterface\ m) \Rightarrow P\Sigma \to m\ P\Sigma$$

$$\begin{aligned}
&mnext\ \varsigma@(Call\ f\ aes, \rho, \sigma, t) = \mathbf{do}\\
&\quad proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow fun\ (\rho, \sigma)\ f\\
&\quad t' \leftarrow tick\ proc\ \varsigma\\
&\quad as \leftarrow mapM\ (alloc\ t')\ vs\\
&\quad ds \leftarrow mapM\ (arg\ \rho)\ aes\\
&\quad \mathbf{let}\ \rho'' = \rho' \mathbin{/\!/} [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as]\\
&\quad sequence\ [a \mapsto d \mid a \leftarrow as \mid d \leftarrow ds]\\
&\quad return\ (call, \rho'', \sigma', t')\\
&mnext\ \varsigma = return\ \varsigma
\end{aligned}$$

Refactoring Plan

- ✓ Capture non-determinism in the monad
- ➔ Pull the store into the monad
- Pull the time into the monad
- Abstract over *k*-CFA addresses

Refactoring Plan

✓ • Capture non-determinism in the monad

✓ • Pull the store into the monad

➔ • Pull the time into the monad

• Abstract over *k*-CFA addresses

Semantic Interface

$$
\begin{aligned}
&\mathbf{class}\ Monad\ m \Rightarrow CPSInterface\ m\ \mathbf{where}\\
&\quad fun\ \ :: Env \to AExp \to m\ Val\\
&\quad arg\ \ :: Env \to AExp \to m\ Val\\
&\quad (\mapsto)\ :: Addr \to Val \to m\ ()\\
&\quad alloc :: Time \to Var \to m\ Addr\\
&\quad tick\ \ :: Val \to P\Sigma \to m\ Time
\end{aligned}
$$

Semantic Interface

$$
\begin{array}{l}
\textbf{class } Monad\ m \Rightarrow CPSInterface\ m\ \textbf{where} \\
\quad \begin{array}{ll}
fun & :: Env \rightarrow AExp \rightarrow m\ Val \\
arg & :: Env \rightarrow AExp \rightarrow m\ Val \\
(\mapsto) & :: Addr \rightarrow Val \rightarrow m\ () \\
alloc & :: Var \rightarrow m\ Addr \\
tick & :: Val \rightarrow P\Sigma \rightarrow m\ ()
\end{array}
\end{array}
$$

Semantic Interface

$$
\begin{array}{l}
\textbf{class } \mathit{Monad}\ m \Rightarrow \mathit{CPSInterface}\ m\ \textbf{where} \\
\quad \mathit{fun} \ :: \mathit{Env} \to \mathit{AExp} \to m\ \mathit{Val} \\
\quad \mathit{arg} \ :: \mathit{Env} \to \mathit{AExp} \to m\ \mathit{Val} \\
\quad (\mapsto) \ :: \mathit{Addr} \to \mathit{Val} \to m\ () \\
\quad \mathit{alloc} :: \mathit{Var} \to m\ \mathit{Addr} \\
\quad \mathit{tick} \ :: \mathit{Val} \to P\Sigma \to m\ ()
\end{array}
$$

$\overbrace{(\mathit{CExp}, \mathit{Env})}$ - "*pure* partial state"

$$mnext :: (CPSInterface\ m) \Rightarrow P\Sigma \rightarrow m\ P\Sigma$$

$$mnext\ \varsigma@(Call\ f\ aes, \rho, \sigma, t) = \mathbf{do}$$
$$proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow fun\ (\rho, \sigma)\ f$$
$$t' \leftarrow tick\ proc\ ps$$
$$as \leftarrow mapM\ (alloc\ t')\ vs$$
$$ds \leftarrow mapM\ (arg\ \rho)\ aes$$
$$\mathbf{let}\ \rho'' = \rho' \mathbin{/\!/} [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as]$$
$$sequence\ [a \mapsto d \mid a \leftarrow as \mid d \leftarrow ds]$$
$$return\ (call, \rho'', \sigma', t')$$

$$mnext\ \varsigma = return\ \varsigma$$

$$mnext :: (CPSInterface\ m) \Rightarrow P\Sigma \rightarrow m\ P\Sigma$$

$$\begin{aligned}
&mnext\ \varsigma@(Call\ f\ \ aes, \rho, \sigma, t) = \mathbf{do} \\
&\quad proc@(Clo\ (vs \Rightarrow call, \rho')) \leftarrow\ fun\ (\rho, \sigma)\ f \\
&\quad tick\ proc\ ps \\
&\quad as \leftarrow mapM\ alloc\ vs \\
&\quad ds \leftarrow mapM\ (arg\ \rho)\ aes \\
&\quad \mathbf{let}\ \rho'' = \rho' \mathbin{/\!/} [\,v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as] \\
&\quad sequence\ [\,a \mapsto d \mid a \leftarrow as \mid d \leftarrow ds] \\
&\quad return\ (call, \rho'', \sigma', t') \\
&mnext\ \varsigma = return\ \varsigma
\end{aligned}$$

Refactoring Plan

- ✓ Capture non-determinism in the monad
- ✓ Pull the store into the monad
- ➔ Pull the time into the monad
- Abstract over *k*-CFA addresses

Refactoring Plan

- ✓ • Capture non-determinism in the monad
- ✓ • Pull the store into the monad
- ✓ • Pull the time into the monad
- → • Abstract over *k*-CFA addresses

$$
\begin{aligned}
\textbf{type } P\Sigma \quad &= \quad (CExp, Env) \\
\textbf{type } Env \quad &= \quad Var \rightharpoonup Addr \\
\textbf{data } Val \quad &= \quad Clo\ (Lambda, Env) \\
\textbf{type } Store \quad &= \quad Addr \rightharpoonup \mathcal{P}(Val)
\end{aligned}
$$

$$
\begin{aligned}
\textbf{type } Addr \quad &= (Var, Time) \\
\textbf{type } Time \quad &= [CExp]
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{type}\ P\Sigma &&= (\mathit{CExp}, \mathit{Env}) \\
&\textbf{type}\ \mathit{Env} &&= \mathit{Var} \rightharpoonup \mathit{Addr} \\
&\textbf{data}\ \mathit{Val} &&= \mathit{Clo}\ (\mathit{Lambda}, \mathit{Env}) \\
&\textbf{type}\ \mathit{Store} &&= \mathit{Addr} \rightharpoonup \mathcal{P}(\mathit{Val})
\end{aligned}
$$

$$
\begin{aligned}
\textbf{type}\ P\Sigma\ a &= (CExp, Env\, a)\\
\textbf{type}\ Env\ a &= Var \rightharpoonup a\\
\textbf{data}\ Val\ a &= Clo\ (Lambda, Env\ a)\\
\textbf{type}\ Store\ a &= a \rightharpoonup \mathcal{P}(Val\ a)
\end{aligned}
$$

Refactoring Plan

✓ • Capture non-determinism in the monad

✓ • Pull the store into the monad

✓ • Pull the time into the monad

→ • Abstract over *k*-CFA addresses

Refactoring Plan

- ✓ Capture non-determinism in the monad
- ✓ Pull the store into the monad
- ✓ Pull the time into the monad
- ✓ Abstract over *k*-CFA addresses

Refactoring Plan

- `List` Capture non-determinism in the monad
- `State` Pull the store into the monad
- `Writer` Pull the time into the monad
- Abstract over *k*-CFA addresses

Monadic Small-Step Transition

$$
\begin{array}{l}
mnext :: CPSInterface\ m\ a \Rightarrow P\Sigma\ a \rightarrow m\ (P\Sigma\ a) \\
mnext\ ps@(Call\ f\ aes, \rho) = \mathbf{do} \\
\quad proc@(Clo\ (vs \Rightarrow call', \rho')) \leftarrow fun\ \rho\ f \\
\quad tick\ proc\ ps \\
\quad as \leftarrow mapM\ alloc\ vs \\
\quad ds \leftarrow mapM\ (arg\ \rho)\ aes \\
\quad \mathbf{let}\ \rho'' = \rho' \mathbin{/\!/} [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as] \\
\quad sequence\ [a \mapsto d \mid a \leftarrow as \mid d \leftarrow ds] \\
\quad return\ (call', \rho'') \\
mnext\ \varsigma = return\ \varsigma
\end{array}
$$

Monadic Small-Step Transition

$$
\begin{array}{l}
mnext :: CPSInterface\ m\ a \Rightarrow P\Sigma\ a \rightarrow m\ (P\Sigma\ a) \\
mnext\ ps@(Call\ f\ aes, \rho) = \mathbf{do} \\
\quad proc@(Clo\ (vs \Rightarrow call', \rho')) \leftarrow fun\ \rho\ f \\
\quad tick\ proc\ ps \\
\quad as \leftarrow mapM\ alloc\ vs \\
\quad ds \leftarrow mapM\ (arg\ \rho)\ aes \\
\quad \mathbf{let}\ \rho'' = \rho' \mathbin{/\!/} [v \Longrightarrow a \mid v \leftarrow vs \mid a \leftarrow as] \\
\quad sequence\ [a \mapsto d \mid a \leftarrow as \mid d \leftarrow ds] \\
\quad return\ (call', \rho'') \\
mnext\ \varsigma = return\ \varsigma
\end{array}
$$

Fixed

Semantic Interface

$$
\begin{aligned}
&\textbf{class}\ Monad\ m \Rightarrow CPSInterface\ m\ a\ \textbf{where}\\
&\quad fun\ \ :: Env\ a \rightarrow AExp \rightarrow m\ (Val\ a)\\
&\quad arg\ \ :: Env\ a \rightarrow AExp \rightarrow m\ (Val\ a)\\
&\quad (\mapsto)\ :: a \rightarrow Val\ a \rightarrow m\ ()\\
&\quad alloc :: Var \rightarrow m\ a\\
&\quad tick\ \ :: Val\ a \rightarrow P\Sigma\ a \rightarrow m\ ()
\end{aligned}
$$

The Semantic Interface

$$
\begin{aligned}
&\mathbf{class}\ Monad\ m \Rightarrow CPSInterface\ m\ a\ \mathbf{where}\\
&\quad fun \ \ :: Env\ a \to AExp \to m\ (Val\ a)\\
&\quad arg \ \ :: Env\ a \to AExp \to m\ (Val\ a)\\
&\quad (\mapsto)\ :: a \to Val\ a \to m\ ()\\
&\quad alloc :: Var \to m\ a\\
&\quad tick \ \ :: Val\ a \to P\Sigma\ a \to m\ ()
\end{aligned}
$$

The Semantic Interface

$$
\begin{aligned}
&\textbf{class } \mathit{Monad}\ m \Rightarrow \mathit{CPSInterface}\ m\ a\ \textbf{where}\\
&\quad \mathit{fun}\ \ :: \mathit{Env}\ a \to \mathit{AExp} \to m\ (\mathit{Val}\ a)\\
&\quad \mathit{arg}\ \ :: \mathit{Env}\ a \to \mathit{AExp} \to m\ (\mathit{Val}\ a)\\
&\quad (\mapsto)\ :: a \to \mathit{Val}\ a \to m\ ()\\
&\quad \mathit{alloc} :: \mathit{Var} \to m\ a\\
&\quad \mathit{tick}\ \ :: \mathit{Val}\ a \to P\Sigma\ a \to m\ ()
\end{aligned}
$$

Needs to be instantiated

So what now?

Instantiating Monadic Semantics

Instance I: Shallow Concrete Interpreter

Instance I: Shallow Concrete Interpreter

`IO`

+

Semantic Interface Implementation

+

Standard driver loop machinery

Addresses

$$\mathbf{data}\ IOAddr = IOAddr\ \{lookup :: IORef\ (Val\ IOAddr)\}$$

Read / Write

$$readIOAddr :: IOAddr \to IO\ (Val\ IOAddr)$$
$$readIOAddr = readIORef \circ lookup$$

$$writeIOAddr :: IOAddr \to Val\ IOAddr \to IO\ ()$$
$$writeIOAddr = writeIORef \circ lookup$$

Semantic Functions for Concrete Semantics

$$
\begin{array}{l}
\textbf{instance}\ \mathit{CPSInterface\ IO\ IOAddr}\ \textbf{where} \\
\quad \mathit{fun}\ \rho\ (\mathit{Lam}\ l) = \mathit{return}\ \$\ \mathit{Clo}\ (l, \rho) \\
\quad \mathit{fun}\ \rho\ (\mathit{Ref}\ \ v)\ = \mathit{readIOAddr}\ (\rho\ !\ v) \\
\quad \mathit{arg}\ \rho\ (\mathit{Lam}\ l) = \mathit{return}\ \$\ \mathit{Clo}\ (l, \rho) \\
\quad \mathit{arg}\ \rho\ (\mathit{Ref}\ \ v)\ = \mathit{readIOAddr}\ (\rho\ !\ v) \\
\quad \mathit{addr} \mapsto v \qquad = \mathit{writeIOAddr}\ \mathit{addr}\ v \\
\quad \mathit{alloc}\ v \qquad\quad = \mathit{liftM}\ \ \mathit{IOAddr}\ \$\ \mathit{newIORef}\ \bot \\
\quad \mathit{tick}\ \_\ \_ \qquad\quad = \mathit{return}\ ()
\end{array}
$$

Semantic Functions for Concrete Semantics

$$\overbrace{}^{Monad}$$

$$
\begin{aligned}
&\textbf{instance } CPSInterface\ \overbrace{IO}\ IOAddr\ \textbf{where} \\
&\quad fun\ \rho\ (Lam\ l) = return\ \$\ Clo\ (l, \rho) \\
&\quad fun\ \rho\ (Ref\ \ v) = readIOAddr\ (\rho\ !\ v) \\
&\quad arg\ \rho\ (Lam\ l) = return\ \$\ Clo\ (l, \rho) \\
&\quad arg\ \rho\ (Ref\ \ v) = readIOAddr\ (\rho\ !\ v) \\
&\quad addr \mapsto v \qquad\quad = writeIOAddr\ \ addr\ \ v \\
&\quad alloc\ v \qquad\qquad = liftM\ \ IOAddr\ \$\ newIORef\ \bot \\
&\quad tick\ \_\ \_ \qquad\quad\ = return\ ()
\end{aligned}
$$

Semantic Functions for Concrete Semantics

$$
\begin{aligned}
&\textbf{instance } \mathit{CPSInterface}\ \overbrace{\mathit{IO}}^{\mathit{Monad}}\ \overbrace{\mathit{IOAddr}}^{\mathit{Addr}}\ \textbf{where} \\
&\quad \mathit{fun}\ \rho\ (\mathit{Lam}\ l) = \mathit{return}\ \$\ \mathit{Clo}\ (l, \rho) \\
&\quad \mathit{fun}\ \rho\ (\mathit{Ref}\ \ v) = \mathit{readIOAddr}\ (\rho\ !\ v) \\
&\quad \mathit{arg}\ \rho\ (\mathit{Lam}\ l) = \mathit{return}\ \$\ \mathit{Clo}\ (l, \rho) \\
&\quad \mathit{arg}\ \rho\ (\mathit{Ref}\ \ v) = \mathit{readIOAddr}\ (\rho\ !\ v) \\
&\quad \mathit{addr} \mapsto v \qquad\quad\ \ = \mathit{writeIOAddr}\ \ \mathit{addr}\ \ v \\
&\quad \mathit{alloc}\ v \qquad\qquad\ \ = \mathit{liftM}\ \ \mathit{IOAddr}\ \$\ \mathit{newIORef}\ \bot \\
&\quad \mathit{tick}\ \_\ \_ \qquad\qquad\ = \mathit{return}\ ()
\end{aligned}
$$

Driver Loop

$$
\begin{aligned}
&interpret :: CExp \to IO\ (P\Sigma\ IOAddr) \\
&interpret\ e = go\ (e, Map.empty) \\
&\quad \textbf{where}\ go :: (P\Sigma\ IOAddr) \to IO\ (P\Sigma\ IOAddr) \\
&\qquad\qquad go\ s = \textbf{do}\ s' \leftarrow mnext\ s \\
&\qquad\qquad\qquad\qquad \textbf{case}\ s'\ \textbf{of}\ \begin{array}{ll} x@(Exit, \_) & \to return\ x \\ y & \to go\ y \end{array}
\end{aligned}
$$

Driver Loop

$$
\begin{aligned}
&interpret :: CExp \rightarrow IO\ (P\Sigma\ IOAddr) \\
&interpret\ e = go\ \boxed{(e, Map.empty)}\ s_0 \\
&\quad \textbf{where}\ go :: (P\Sigma\ IOAddr) \rightarrow IO\ (P\Sigma\ IOAddr) \\
&\qquad\quad go\ s = \textbf{do}\ s' \leftarrow mnext\ s \\
&\qquad\qquad\qquad \textbf{case}\ s'\ \textbf{of}\ x@(Exit, \_) \rightarrow return\ x \\
&\qquad\qquad\qquad\qquad\qquad\ y \qquad\qquad\quad \rightarrow go\ y
\end{aligned}
$$

Driver Loop

$$
\begin{aligned}
&interpret :: CExp \to IO\ (P\Sigma\ IOAddr)\\
&interpret\ e = go\ (e, Map.empty)\\
&\quad \textbf{where}\ go :: (P\Sigma\ IOAddr) \to IO\ (P\Sigma\ IOAddr)\\
&\qquad\qquad go\ s = \textbf{do}\ s' \leftarrow mnext\ s\\
&\qquad\qquad\qquad\qquad \textbf{case}\ s'\ \textbf{of}\ x@(Exit, \_) \to return\ x\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad y \qquad\qquad\quad \to go\ y
\end{aligned}
$$

s_0

Driver Loop

$$
\begin{aligned}
&\mathit{interpret} :: \mathit{CExp} \to \mathit{IO}\ (P\Sigma\ \mathit{IOAddr})\\
&\mathit{interpret}\ e = \mathit{go}\ \boxed{(e, \mathit{Map.empty})}\\
&\quad \textbf{where}\ \mathit{go} :: (P\Sigma\ \mathit{IOAddr}) \to \mathit{IO}\ (P\Sigma\ \mathit{IOAddr})\\
&\qquad\quad \mathit{go}\ s = \textbf{do}\ s' \leftarrow \boxed{\mathit{mnext}}\ s\\
&\qquad\qquad\qquad\quad \textbf{case}\ s'\ \textbf{of}\ x@(\mathit{Exit}, \_) \to \mathit{return}\ x\\
&\qquad\qquad\qquad\qquad\qquad\quad y \qquad\qquad\quad\ \to \mathit{go}\ y
\end{aligned}
$$

$s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_4 \rightarrow \ldots$

Instance II: Collecting Abstract Interpreter

Instance II: Collecting Abstract Interpreter

`State(State(List)) Monad`

+

Semantic Interface Implementation

+

Generic fixed point machinery

Collecting Semantics and Fixed Points

$$\hat{f} \in \mathcal{P}(\Sigma) \rightarrow \mathcal{P}(\Sigma)$$

$$\hat{f}(\hat{S}) = \{\hat{\varsigma}_0\} \cup \{\hat{\varsigma}' \mid \hat{\varsigma} \leadsto \hat{\varsigma}' \text{ and } \hat{\varsigma} \in \hat{S}\}$$

$$\mathrm{lfp}_{\sqsubseteq} f = \bigsqcup_{i \geq 0} f^i(\bot)$$

$$\mathrm{lfp}_{\sqsubseteq}\, f = \bigsqcup_{i \geq 0} f^{i}(\bot)$$

$$\begin{array}{l}
\mathit{kleeneIt} :: (\mathit{Lattice}\ a) \Rightarrow (a \to a) \to a \\
\mathit{kleeneIt}\ f = \mathit{loop}\ \bot \\
\quad \textbf{where}\ \mathit{loop}\ c = \textbf{let}\ c' = f\ c\ \textbf{in} \\
\qquad\qquad\qquad\qquad \textbf{if}\ c' \sqsubseteq c\ \textbf{then}\ c\ \textbf{else}\ \mathit{loop}\ c'
\end{array}$$

$$\mathrm{lfp}_{\sqsubseteq}\, f = \bigsqcup_{i \geq 0} f^i(\bot)$$

$$\begin{aligned}
&\mathit{kleeneIt} :: (\mathit{Lattice}\ a) \Rightarrow (a \to a) \to a \\
&\mathit{kleeneIt}\ f = \mathit{loop}\ \bot \\
&\quad \mathbf{where}\ \mathit{loop}\ c = \mathbf{let}\ c' = f\ c\ \mathbf{in} \\
&\qquad\qquad\qquad\qquad \mathbf{if}\ c' \sqsubseteq c\ \mathbf{then}\ c\ \mathbf{else}\ \mathit{loop}\ c'
\end{aligned}$$

$$\begin{aligned}
&\mathbf{class}\ \mathit{Collecting}\ m\ a\ \mathit{fp} \mid \mathit{fp} \to a, \mathit{fp} \to m\ \mathbf{where} \\
&\quad \mathit{applyStep} :: (a \to m\ a) \to \mathit{fp} \to \mathit{fp} \\
&\quad \mathit{inject} :: a \to \mathit{fp}
\end{aligned}$$

$$\begin{aligned}
&\mathit{exploreFP} :: (\mathit{Lattice}\ \mathit{fp}, \mathit{Collecting}\ m\ a\ \mathit{fp}) \Rightarrow \\
&\qquad\qquad\quad (a \to m\ a) \to a \to \mathit{fp} \\
&\mathit{exploreFP}\ \mathit{step}\ c = \mathit{kleeneIt}\ \mathcal{F} \\
&\quad \mathbf{where}\ \mathcal{F}\ s = \mathit{inject}\ c \sqcup \mathit{applyStep}\ \mathit{step}\ s
\end{aligned}$$

$$\mathrm{lfp}_{\sqsubseteq}\, f = \bigsqcup_{i \geq 0} f^{i}(\bot)$$

$$\begin{aligned}
&\mathit{kleeneIt} :: (\mathit{Lattice}\ a) \Rightarrow (a \to a) \to a\\
&\mathit{kleeneIt}\ f\ = \mathit{loop}\ \bot\\
&\quad \mathbf{where}\ \mathit{loop}\ c = \mathbf{let}\ c' = f\ c\ \mathbf{in}\\
&\qquad\qquad\qquad\quad \mathbf{if}\ c' \sqsubseteq c\ \mathbf{then}\ c\ \mathbf{else}\ \mathit{loop}\ c'
\end{aligned}$$

$(\rightsquigarrow)$ →

$$\begin{aligned}
&\mathbf{class}\ \mathit{Collecting}\ m\ a\ \mathit{fp} \mid \mathit{fp} \to a, \mathit{fp} \to m\ \mathbf{where}\\
&\quad \mathit{applyStep} :: (a \to m\ a) \to \mathit{fp} \to \mathit{fp}\\
&\quad \mathit{inject} :: a \to \mathit{fp}
\end{aligned}$$

$$\begin{aligned}
&\mathit{exploreFP} :: (\mathit{Lattice}\ \mathit{fp}, \mathit{Collecting}\ m\ a\ \mathit{fp}) \Rightarrow\\
&\qquad\qquad\quad (a \to m\ a) \to a \to \mathit{fp}\\
&\mathit{exploreFP}\ \mathit{step}\ c = \mathit{kleeneIt}\ \mathcal{F}\\
&\quad \mathbf{where}\ \mathcal{F}\ s = \mathit{inject}\ c \sqcup \mathit{applyStep}\ \mathit{step}\ s
\end{aligned}$$

$$\mathrm{lfp}_{\sqsubseteq} f = \bigsqcup_{i \geq 0} f^i(\bot)$$

$$\begin{array}{l}
\mathit{kleeneIt} :: (\mathit{Lattice}\ a) \Rightarrow (a \to a) \to a \\
\mathit{kleeneIt}\ f = \mathit{loop}\ \bot \\
\quad \textbf{where}\ \mathit{loop}\ c = \textbf{let}\ c' = f\ c\ \textbf{in} \\
\qquad\qquad\qquad\qquad \textbf{if}\ c' \sqsubseteq c\ \textbf{then}\ c\ \textbf{else}\ \mathit{loop}\ c'
\end{array}$$

$(\rightsquigarrow)$ → *applyStep*

$\{\cdot\}$ → *inject*

$$\begin{array}{l}
\textbf{class}\ \mathit{Collecting}\ m\ a\ \mathit{fp} \mid \mathit{fp} \to a, \mathit{fp} \to m\ \textbf{where} \\
\quad \mathit{applyStep} :: (a \to m\ a) \to \mathit{fp} \to \mathit{fp} \\
\quad \mathit{inject} :: a \to \mathit{fp}
\end{array}$$

$$\begin{array}{l}
\mathit{exploreFP} :: (\mathit{Lattice}\ \mathit{fp}, \mathit{Collecting}\ m\ a\ \mathit{fp}) \Rightarrow \\
\qquad\qquad\quad (a \to m\ a) \to a \to \mathit{fp} \\
\mathit{exploreFP}\ \mathit{step}\ c = \mathit{kleeneIt}\ \mathcal{F} \\
\quad \textbf{where}\ \mathcal{F}\ s = \mathit{inject}\ c \sqcup \mathit{applyStep}\ \mathit{step}\ s
\end{array}$$

$$\mathrm{lfp}_{\sqsubseteq}\, f = \bigsqcup_{i \geq 0} f^{i}(\bot)$$

$$
\begin{array}{l}
\mathit{kleeneIt} :: (\mathit{Lattice}\ a) \Rightarrow (a \to a) \to a \\
\mathit{kleeneIt}\ f = \mathit{loop}\ \bot \\
\quad \textbf{where}\ \mathit{loop}\ c = \textbf{let}\ c' = f\ c\ \textbf{in} \\
\qquad\qquad\qquad\qquad \textbf{if}\ c' \sqsubseteq c\ \textbf{then}\ c\ \textbf{else}\ \mathit{loop}\ c'
\end{array}
$$

$$
\begin{array}{l}
\textbf{class}\ \mathit{Collecting}\ m\ a\ \mathit{fp} \mid \mathit{fp} \to a, \mathit{fp} \to m\ \textbf{where} \\
\mathit{applyStep} :: (a \to m\ a) \to \mathit{fp} \to \mathit{fp} \\
\mathit{inject} :: a \to \mathit{fp}
\end{array}
$$

$(\leadsto)$ → $\mathit{applyStep}$

$\{\cdot\}$ → inject

$$
\begin{array}{l}
\mathit{exploreFP} :: (\mathit{Lattice}\ \mathit{fp}, \mathit{Collecting}\ m\ a\ \mathit{fp}) \Rightarrow \\
\qquad\qquad (a \to m\ a) \to a \to \mathit{fp} \\
\mathit{exploreFP}\ \mathit{step}\ c = \mathit{kleeneIt}\ \mathcal{F} \\
\quad \textbf{where}\ \mathcal{F}\ s = \underbrace{\mathit{inject}\ c \sqcup \mathit{applyStep}\ \mathit{step}\ s}_{\hat{f}(\hat{S}) = \{\hat{\varsigma}_0\} \cup \{\hat{\varsigma}' \mid \hat{\varsigma} \leadsto \hat{\varsigma}' \text{ and } \hat{\varsigma} \in \hat{S}\}}
\end{array}
$$

$$
\begin{aligned}
runAnalysis &:: (CPSInterface\ m\ a, Lattice\ fp,\\
&\qquad Collecting\ m\ (P\Sigma\ a)\ fp) \Rightarrow\\
&\qquad CExp \rightarrow fp\\
runAnalysis\ e &= exploreFP\ mnext\ (e, Map.empty)
\end{aligned}
$$

$$
\begin{aligned}
runAnalysis :: (&CPSInterface\ m\ a, Lattice\ fp,\\
&Collecting\ m\ (P\Sigma\ a)\ fp) \Rightarrow\\
&CExp \rightarrow fp
\end{aligned}
$$

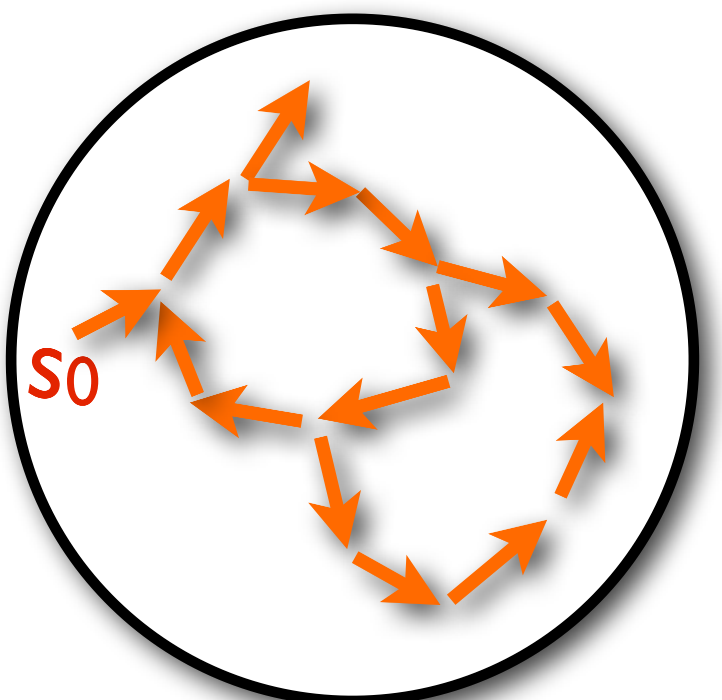
$$
runAnalysis\ e = exploreFP\ mnext\ \underbrace{\boxed{(e, Map.empty)}}_{s_0}
$$

$$runAnalysis :: (CPSInterface\ m\ a, Lattice\ fp,$$
$$Collecting\ m\ (P\Sigma\ a)\ fp) \Rightarrow$$
$$CExp \rightarrow fp$$
$$runAnalysis\ e = exploreFP\ mnext\ \boxed{(e, Map.empty)}$$

s_0

$$
\begin{aligned}
runAnalysis :: (&CPSInterface\ m\ a, Lattice\ fp,\\
&Collecting\ m\ (P\Sigma\ a)\ fp) \Rightarrow\\
&CExp \rightarrow fp
\end{aligned}
$$

$$runAnalysis\ e = exploreFP\ mnext\ \boxed{(e, Map.empty)}$$



Implementing
Collecting Abstract Interpreter
in 3 steps

1. Fixing the Monad

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2. Providing Denotations

$$
\begin{array}{l}
\textbf{instance } \mathit{CPSInterface} \\
\quad (\mathit{StorePassing}\ (\mathit{Store}\ \mathit{Integer})\ \mathit{Integer})\ \mathit{Integer} \\
\quad \textbf{where} \\
\qquad \mathit{fun}\ \rho\ (\mathit{Lam}\ l) = \mathit{return}\ \$\ \mathit{Clo}\ (l, \rho) \\
\qquad \mathit{fun}\ \rho\ (\mathit{Ref}\ v) = \mathit{lift}\ \$\ \mathit{getsNDSet}\ \$\ \lambda\sigma \to \sigma\,!\,(\rho\,!\,v) \\
\qquad \mathit{arg}\ \rho\ (\mathit{Lam}\ l) = \mathit{return}\ \$\ \mathit{Clo}\ (l, \rho) \\
\qquad \mathit{arg}\ \rho\ (\mathit{Ref}\ v) = \mathit{lift}\ \$\ \mathit{getsNDSet}\ \$\ \lambda\sigma \to \sigma\,!\,(\rho\,!\,v) \\
\qquad a \mapsto d = \mathit{lift}\ \$\ \mathit{modify}\ \$ \\
\qquad\qquad \mathit{Map.insert}\ a\ (\mathit{singleton}\ d) \\
\qquad \mathit{alloc}\ v = \mathit{gets}\ \mathit{id} \\
\qquad \mathit{tick}\ \mathit{proc}\ \mathit{ps} = \mathit{modify}\ \$\ \lambda t \to t + 1
\end{array}
$$

3. Starting and Stepping

$$
\begin{array}{l}
\textbf{instance } (Ord\ s, Ord\ a, Ord\ g, HasInitial\ g, Lattice\ s) \Rightarrow \\
\qquad\qquad Collecting\ (StorePassing\ s\ g) \\
\qquad\qquad\qquad\qquad (P\Sigma\ a) \\
\qquad\qquad\qquad\qquad (\mathcal{P}\ ((P\Sigma\ a, g), s))\ \textbf{where} \\
\quad inject\ p = singleton\ \$\ ((p, initial), \bot) \\
\quad applyStep\ step\ fp = joinWith\ runStep\ fp\ \textbf{where} \\
\qquad runStep\ ((\varsigma, t), s) = \\
\qquad\quad Set.fromList\ \$\ runStateT\ (runStateT\ (step\ \varsigma)\ t)\ s
\end{array}
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\quad \mathit{applyStep}\ \mathit{step}\ \mathit{fp} = \mathit{joinWith}\ \mathit{runStep}\ \mathit{fp}\ \textbf{where} \\
\qquad \mathit{runStep}\ ((\varsigma, t), s) = \\
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starting

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starting

stepping

$$runAnalysis\ exp :: \mathcal{P}\ ((P\Sigma\ Integer, Integer), Store\ Integer)$$

a program

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$$\underbrace{\qquad\qquad\qquad\qquad}_{(P\Sigma\ Addr) \times Time \times (Store\ Addr)}$$

$$\text{where } Addr = Integer$$

$$Time = Integer$$

Abstract Interpretation
can be seen as a computational effect

Monadic refactoring disentangles
transitions from their denotation

A monad specifies the state-space

Check the paper

- Generic implementation of polyvariance
- Language-independent store
- Language-independent abstract counting
- Reusable abstract garbage collection
- Pluggable widening strategies

Try the code

`http://github.com/ilyasergey/monadic-cfa`

- Featherweight Java
- Direct-style λ-calculus
- Monadic machinery
- Full-fledged abstract GC
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- Lots of examples

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Thanks